

ON A SUFFICIENT CONDITION FOR PROXIMITY

BY

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ABSTRACT. A closed subspace M in a Banach space X is called U -proximal if it satisfies: $(1 + \rho)S \cap (S + M) \subseteq S + \varepsilon(\rho)(S \cap M)$, for some positive valued function $\varepsilon(\rho)$, $\rho > 0$, and $\varepsilon(\rho) \rightarrow 0$ as $\rho \rightarrow 0$, where S is the closed unit ball of X . One of the important properties of this class of subspaces is that the metric projections are continuous. We show that many interesting subspaces are U -proximal, for example, the subspaces with the 2-ball property (semi M -ideals) and certain subspaces of compact operators in the spaces of bounded linear operators.

1. Introduction. We call a closed subspace M of a real Banach space X an M -ideal if the annihilator M^\perp of M is an L -summand in X^* . This notion was formulated and studied by Alfsen and Effros [1]. It was proved that if M is an M -ideal, then M is a proximal subspace of X [1], [5]. In [4], Hennessee showed that the space of compact operators on l^p (or c_0), $1 < p < \infty$, is an M -ideal in the space of bounded linear operators on l^p (or c_0 respectively). This theorem is also true for operators from l^p into l^q , $1 < p \leq q < \infty$ [11]. (It is well known that if $1 \leq q < p < \infty$, then every bounded linear operator from l^p into l^q is compact.) M -ideal theory provides a convenient tool to study the approximation of operators by the space of compact operators and has been investigated by many authors [3], [4], [5], [9], [11], [15], [16]. However, in some cases, the class of M -ideals appears to be too restricted; for example, the space of compact operators on l^1 is not an M -ideal in the space of bounded linear operators on l^1 [16]. It is our attempt to consider another sufficient condition for proximity which preserves certain important properties of M -ideals and also includes some other interesting classes of proximal subspaces.

Motivated by a lemma of Holmes in [5], we call a closed subspace M of a Banach space X U -proximal if there exists a positive function $\varepsilon(\rho)$, $\rho > 0$, with $\varepsilon(\rho) \rightarrow 0$ as $\rho \rightarrow 0$ and satisfies

$$(1 + \rho)S \cap (S + M) \subseteq S + \varepsilon(\rho)(S \cap M), \quad \rho > 0,$$

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where S denotes the closed unit ball of X . Examples of U -proximal subspaces are:

- (i) X is uniformly convex and M is a closed subspace of X ;
- (ii) $X = B(K)$, the space of bounded functions on a topological space K and $M = C(K)$, the space of bounded continuous functions on K ;
- (iii) M is a semi M -ideal or semi L -summand in a Banach space X ;
- (iv) $X = L^\infty(\Omega, L^1)$, the space of bounded Bochner measurable functions from a σ -finite measure space Ω into L^1 and M is the subspace of f in X such that $f(\Omega)$ is weakly precompact. In particular, if we let $L(E, F)$ ($K(E, F)$) denote the space of bounded linear operators (compact operators, respectively), then $K(L^1(\Omega), l^1)$ is proximal in $L(L^1(\Omega), l^1)$.

Our paper is divided into six sections. In §2, we define some basic terminologies and give several reformulations of the definition of U -proximity. The metric projection from X into a proximal subspace M is the map P which sends $x \in X$ to the set of best approximations from M to x . The study of the continuity of metric projections is an important component of the theory of best approximation. In §3, we show that if M is U -proximal, then the metric projection P is continuous (with respect to the Hausdorff metric in the range). We also give a condition for P to be Lipschitz continuous. In §§4–6, we show that the examples listed above are in fact U -proximal subspaces.

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2. Definitions and preliminaries. Let X be a real Banach space, let $S_r(X)$ (or S_r) denote the closed ball of radius r centered at 0; $S_1(X) = S(X)$ ($S_1 = S$). If M is a closed subspace of X , then for $x \in X$, we denote the subspace generated by M and x by $\langle M, x \rangle$, and denote $S_r(\langle M, x \rangle)$ by $S_r[x]$.

Let $F(X)$ be a family of nonempty bounded closed subsets of X . For any $A, B \in F(X)$, we define

$$d_H(A, B) = \inf\{r: A \subseteq B + S_r \text{ and } B \subseteq A + S_r\}.$$

Then d_H is a metric on $F(X)$ and is called the *Hausdorff metric*.

For $x \in X$ and for any subset A in X , we define another distance function $d(x, A) = \inf\{\|x - z\|: z \in A\}$. A point y in a closed subspace M of X is called a *best approximation* from M to x if $\|x - y\| = d(x, M)$. M is called a *proximal subspace* of X if every $x \in X$ has a best approximation from M . Propositions 2.2 and 2.3 will give the motivation of a simple sufficient condition for M to be a proximal subspace.

LEMMA 2.1. Let g be a concave function defined on $[0, 1]$ with $g(0) > 0$ and $g(1) = 0$. Let h be a function defined on $[0, a]$, $a > 1$, with $h(x) = ag(x/a)$, $x \in [0, a]$. Then $h(1) \geq h(x) - g(x)$ for $0 \leq x \leq 1$.

PROOF. Note that the derivatives $h'(x)$, $g'(x)$ exist and decrease almost everywhere. Hence

$$h'(x) - g'(x) = g'\left(\frac{x}{a}\right) - g'(x) \geq 0 \text{ a.e.}$$

and

$$h(1) = h(1) - g(1) \geq h(x) - g(x), \quad 0 \leq x \leq 1. \quad \square$$

Let X be a two dimensional normed linear space. For $\rho > 0$, let L be a line which is tangent to S at w and cuts the sphere $\{z: \|z\| = 1 + \rho\}$ at x and y . Let $(x': y')$ be any open line segment which is parallel to L , with $\|x'\| = 1 + \rho$, $\|y'\| = 1$ and does not intersect S . Lemma 2.1 implies that $\|x' - y'\| \leq \max\{\|x - w\|, \|y - w\|\}$.

Let M be a proximal subspace in X , let $x \in X \setminus M$ and let $X_1 = \langle M, x \rangle$. Let $f \in X_1^*$ be such that $f^{-1}(0) = M$, $\|f\| = 1$ and define

$$\alpha(x, \rho) = d_H((1 + \rho)S[x] \cap f^{-1}(1), S[x] \cap f^{-1}(1)), \quad \rho > 0.$$

It follows from simple geometry that for each $x \in X \setminus M$,

$$|\alpha(x, \rho) - \alpha(x, \rho')| \leq \frac{2|1 + \rho + \rho'|}{\rho} |\rho - \rho'| \text{ for } \rho > 0.$$

Hence $\alpha(x, \cdot)$ is continuous on \mathbf{R}^+ .

PROPOSITION 2.2. Let M be a proximal subspace of a Banach space X and let $\alpha(x, \rho)$ be defined as above. Then

$$\alpha(x, \rho) = \inf\{r > 0: (1 + \rho)S \cap (S[x] + M) \subseteq S + rS(M)\}.$$

PROOF. Without loss of generality, we may assume that $X = \langle M, x \rangle$. It is clear from the definition of $\alpha(x, \rho)$ that

$$\alpha(x, \rho) \leq \inf\{r > 0: (1 + \rho)S \cap (S + M) \subseteq S + rS(M)\}.$$

To prove the reverse inequality, let $0 < \varepsilon < 1$ and let $y \in (1 + \rho)S \cap (S + M)$. We will consider the case $y \notin M$ first. Assume $f(y) > 0$ (the case $f(y) < 0$ is similar), and let $y_1 = ty \in f^{-1}(1)$ for some $t > 0$. There exists $w_1 \in S \cap f^{-1}(1)$ such that

$$\|y_1 - w_1\| \leq d(y_1, S \cap f^{-1}(1)) + \varepsilon.$$

If $\|y_1\| \leq 1 + \rho$, we define $x_1 = y_1$, otherwise we define x_1 to be the point on the line segment of $(y_1 : w_1)$ with $\|x_1\| = 1 + \rho$. In either case, it can be shown that $\|x_1 - w_1\| \leq \alpha(x, \rho) + \varepsilon$. By the remark before the proposition,

we can choose a $z \in f^{-1}(f(y)) \cap \{w: \|w\| = 1\}$ such that $\|y - z\| \leq \alpha(x, \rho) + \varepsilon$. Hence

$$y = z + (y - z) \in S + (\alpha(x, \rho) + \varepsilon)S(M). \tag{1}$$

If $y \in M$, we can choose $y' \in (1 + \rho)S \setminus M$ with $\|y' - y\| < \varepsilon/2(1 + \rho)$ and $z' \in f^{-1}(f(y')) \cap S$ satisfies (1). Let

$$z'' = \|z' - (y' - y)\|^{-1}(z' - (y' - y)).$$

Then z'' is in M since

$$f(z' - (y' - y)) = f(z') - f(y') = 0.$$

That

$$\|y - z''\| \leq \varepsilon + (1 + \varepsilon)(\alpha(x, \rho) + \varepsilon) = \beta$$

implies that

$$y = z'' + (y - z'') \in S + \beta S(M).$$

Since this is also true for the $y \notin M$ as in (1) and since ε is arbitrary, we conclude that

$$\alpha(x, \rho) \geq \inf\{r > 0: (1 + \rho)S \cap (S + M) \subseteq S + rS(M)\}. \quad \square$$

The following proposition is the foundation of this paper; the proof is similar to [5, Lemma 2].

PROPOSITION 2.3. *Let M be a closed subspace of X and let $x \in X \setminus M$. Suppose there exists a function $\varepsilon: \mathbf{R}^+ \rightarrow \mathbf{R}^+$ (depending on x) such that $\varepsilon(\rho) \rightarrow 0$ as $\rho \rightarrow 0$ and*

$$(1 + \rho)S \cap \overline{(S[x] + M)} \subseteq \overline{S + \varepsilon(\rho)S(M)}, \quad \rho > 0.$$

Then x has a best approximation from M .

Furthermore, if $\|x\| \leq 1 + \rho_0$ and if $\inf\{\|x - z\|: z \in M\} \leq 1$. Then given any $r > \varepsilon(\rho_0)$, there exists a best approximation z_0 in M such that $\|z_0\| < r$.

PROOF. Without loss of generality, assume that $\|x\| = 1 + \rho_0$ and $\inf\{\|x - z\|: z \in M\} = 1$. We claim that $x \in \overline{S[x] + M}$. Indeed, a sequence $\{z_n\}$ in M can be chosen such that $\|x - z_n\| \rightarrow 1$. Let

$$x_n = \|x - z_n\|^{-1}(x - z_n) + z_n,$$

then the sequence $\{x_n\}$ is contained in $S[x] + M$ and $x_n \rightarrow x$.

Choose a sequence of positive numbers $\{\rho_n\}$ such that $\rho_n \rightarrow 0$ and $\sum_{n=0}^{\infty} \varepsilon(\rho_n) < r$ where $r > \varepsilon(\rho_0)$. By hypothesis there exists $z_1 \in \varepsilon(\rho_0)S(M)$ with $\|x - z_1\| \leq 1 + \rho_1$. Note that $x - z_1$ is also in $\overline{S[x] + M}$, the same argument yields a z_2 in $\varepsilon(\rho_1)S(M)$ with $\|x - z_1 - z_2\| \leq 1 + \rho_2$. Inductively, we can find a sequence $\{z_n\}$ such that $z_n \in \varepsilon(\rho_{n-1})S(M)$ and $\|x - \sum_{k=1}^n z_k\| \leq 1 + \rho_n$. Let $z_0 = \sum_{n=1}^{\infty} z_n$, then $\|z_0\| \leq \sum_{n=0}^{\infty} \varepsilon(\rho_n) < r$ and $\|x - z_0\| = 1$. This completes the proof. \square

PROPOSITION 2.4. *Let M be a closed subspace in X and let $x \in X \setminus M$. Then the following conditions are equivalent:*

(i) *there exists an $\epsilon: \mathbf{R}^+ \rightarrow \mathbf{R}^+$ such that $\epsilon(\rho) \rightarrow 0$ as $\rho \rightarrow 0$ and*

$$(1 + \rho)S \cap \overline{(S[x] + M)} \subseteq \overline{S + \epsilon(\rho)S(M)}, \quad \rho > 0.$$

(ii) *there exists an $\epsilon': \mathbf{R}^+ \rightarrow \mathbf{R}^+$ such that $\epsilon'(\rho) \rightarrow 0$ as $\rho \rightarrow 0$ and*

$$(1 + \rho)S \cap (S[x] + M) \subseteq S + \epsilon'(\rho)S(M), \quad \rho > 0.$$

(iii) *there exists an $\tilde{\epsilon}: \mathbf{R}^+ \rightarrow \mathbf{R}^+$ such that $\tilde{\epsilon}$ is increasing, continuous, $\tilde{\epsilon}(\rho) \rightarrow 0$ as $\rho \rightarrow 0$ and*

$$(1 + \rho)S \cap (S[x] + M) \subseteq S + \tilde{\epsilon}(\rho)S(M), \quad \rho > 0.$$

Moreover, if one of the three conditions holds, the function $\alpha(x, \rho)$ defined in Proposition 2.2 converges to zero as $\rho \rightarrow 0$.

PROOF. (i) \Rightarrow (ii). Let $y \in (1 + \rho)S \cap (S[x] + M)$. By Proposition 2.3 if $r = 2\epsilon(\rho)$, then there exists a $z \in M$ such that $\|y - z\| \leq 1$ and $\|z\| < 2\epsilon(\rho)$. Let $\epsilon'(\rho) = 2\epsilon(\rho)$, then $y \in S + \epsilon'(\rho)S(M)$ and (ii) follows. (ii) \Rightarrow (i) is clear. To prove (ii) \Rightarrow (iii), we can take $\tilde{\epsilon}(\rho) = \alpha(x, \rho) + \rho$, then $\tilde{\epsilon}$ is an increasing, continuous function. Proposition 2.2 implies that $\alpha(x, \rho) \leq \tilde{\epsilon}(\rho)$. Hence $\tilde{\epsilon}(\rho) \rightarrow 0$ as $\rho \rightarrow 0$. That (iii) \Rightarrow (ii) is obvious. \square

DEFINITION 2.5. *Let M be a closed subspace of a Banach space X , we say that M is locally U -proximal if there exists a function $\epsilon: (X \setminus M) \times \mathbf{R}^+ \rightarrow \mathbf{R}^+$ such that for each fixed x , $\epsilon(x, \cdot)$ is continuous, increasing on ρ , $\epsilon(x, \rho) \rightarrow 0$ as $\rho \rightarrow 0$ and*

$$(1 + \rho)S \cap (S[x] + M) \subseteq S + \epsilon(x, \rho)S(M), \quad x \in M, \quad \rho > 0.$$

M is called U -proximal if the function ϵ can be chosen independent of x and

$$(1 + \rho)S \cap (S + M) \subseteq S + \epsilon(\rho)S(M).$$

It follows from Proposition 2.3 that locally U -proximal and U -proximal subspaces are proximal.

3. Metric projections. Let M be a proximal subspace of X , for each $x \in X$, we define $P(x)$ to be the set of best approximations from M to x . We call the map P from X into $F(M)$, the family of bounded closed subsets in M , as the *metric projection* from X into M .

LEMMA 3.1. *Let M be a locally U -proximal subspace in X . For each $x \in X \setminus M$, let $r_x = d(x, M)$. Then*

$$M \cap (S_{(1+\rho)r_x} + x) \subseteq P(x) + r_x\epsilon(x, \rho)S(M), \quad \rho > 0.$$

PROOF. Without loss of generality, we assume that $r_x = 1$. For $y \in M \cap (S_{(1+\rho)} + x)$, it follows that $\|x - y\| \leq 1 + \rho$. That

$$x - y = (x - z) + (z - y)$$

where z is the best approximation of x , implies that $x - y \in S + M$. Hence

$$(x - y) \in (1 + \rho)S \cap (S + M).$$

Since M is locally U -proximal, $x - y = w + v$ for some w in S and v in $\varepsilon(x, \rho)S(M)$. Note that $(x - w)$ is in M and, in fact, a best approximation to x (for $1 \leq \|x - (x - w)\| = \|w\| \leq 1$). Hence

$$y = (x - w) - v \in P(x) + \varepsilon(x, \rho)S(M). \quad \square$$

LEMMA 3.2. *Let M be a locally U -proximal subspace in X . Then for any $x, y \in X \setminus M$ with $\|x - y\| < \rho$,*

$$d_H(P(x), P(y)) \leq \max \left\{ r_x \varepsilon \left(x, \frac{2\rho}{r_x} \right), r_y \varepsilon \left(y, \frac{2\rho}{r_y} \right) \right\}.$$

PROOF. Since $r_y \leq r_x + \rho$ and

$$(S_{r_y} + y) \subseteq (S_{r_x + 2\rho} + x),$$

it follows that

$$P(y) \subseteq M \cap (S_{r_x + 2\rho} + x).$$

By Lemma 3.1, we have

$$P(y) \subseteq P(x) + r_x \varepsilon \left(x, \frac{2\rho}{r_x} \right) S(M).$$

A similar argument yields that

$$P(x) \subseteq P(y) + r_y \varepsilon \left(y, \frac{2\rho}{r_y} \right) S(M),$$

and the lemma follows easily from these inclusions and the definition of the Hausdorff metric on $F(M)$. \square

THEOREM 3.3. *Let M be a locally U -proximal subspace in X . Suppose for each $\rho > 0$, $\varepsilon(\cdot, \rho)$ is an upper semicontinuous function on $X \setminus M$. Then P is a continuous function from X into $F(M)$.*

PROOF. It is easy to show that P is continuous for $x \in M$. Let $x \in X \setminus M$, for any $\delta > 0$, there exists an η in $(0, 1)$ such that $\varepsilon(x, \eta) < \delta/2$. Since $\varepsilon(\cdot, \eta)$ is upper semicontinuous, there exists a ρ , $0 < \rho < (\eta \cdot r_x)/4$, such that for $\|x - y\| \leq \rho$, $\varepsilon(y, \eta) < \varepsilon(x, \eta) + \delta/2$. Hence for $\|x - y\| \leq \rho$, we have

$$\begin{aligned} d_H(P(x), P(y)) &\leq \max \left\{ r_x \varepsilon \left(x, \frac{2\rho}{r_x} \right), r_y \varepsilon \left(y, \frac{2\rho}{r_y} \right) \right\} \\ &\leq 2r_x \max \left\{ \varepsilon \left(x, \frac{2\rho}{r_x} \right), \varepsilon \left(y, \frac{4\rho}{r_x} \right) \right\} \\ &\leq 2r_x \delta. \end{aligned}$$

This shows that P is continuous at x . \square

THEOREM 3.4. *Let M be a U -proximal subspace of X . Then the metric projection $P: X \rightarrow F(M)$ is continuous.*

PROOF. In this case, we have $\varepsilon(x, \rho) = \varepsilon(y, \rho) = \varepsilon(\rho)$ for any $x, y \in X \setminus M$. Theorem 3.3 implies that P is continuous. \square

COROLLARY 3.5. *Let M be a U -proximal subspace of X . Then the metric projection P admits a continuous selection $s: X \rightarrow M$ (i.e., $s(x) \in P(x)$).*

The following also follows easily from Lemma 3.2:

THEOREM 3.6. *Let M be a subspace of X . Suppose there exists a $k > 0$ such that*

$$(1 + \rho)S \cap (S + M) \subseteq S + k\rho S(M), \quad \rho > 0.$$

Then the metric projection $P: X \rightarrow F(M)$ is a Lipschitz continuous function with Lipschitz constant not greater than $2k$.

In the next three sections, we will see that many interesting examples will satisfy the above inclusion. We remark that the converse of the theorem is not true; for example, let M be a closed subspace of a Hilbert space X , then the metric projection $P: X \rightarrow M$ is a Lipschitz function but $\varepsilon(\rho) \geq \sqrt{\rho^2 + 2\rho}$. We also remark that a U -proximal subspace may not have a uniformly continuous metric projection. Examples (certain closed subspaces in some uniformly convex spaces) can be found in [7], [14]. In general, a set valued function which satisfies the Lipschitz condition does not admit Lipschitz selection; it will be interesting to investigate this question for the metric projections in Theorem 3.6.

4. Some examples. In this section, we will give some simple examples of locally U -proximal and U -proximal subspaces.

PROPOSITION 4.1. *Let M be a finite dimensional subspace in a Banach space X . Then M is locally U -proximal.*

PROOF. Let $x \in X \setminus M$ and assume that $X = \langle M, x \rangle$. Let $\varepsilon(x, \rho) = \alpha(x, \rho) + \rho$, then it follows from the compactness of the unit ball that $\varepsilon(x, \rho) \rightarrow 0$ as $\rho \rightarrow 0$ and by Proposition 2.2,

$$(1 + \rho)S \cap (S + M) \subseteq S + \varepsilon(x, \rho)S(M). \quad \square$$

A Banach space X is called *locally uniformly convex* if for each $x \in X$ with $\|x\| = 1$ and for any $\eta > 0$, there exists a $\delta > 0$ such that for any $y \in X$ with $\|y\| \leq 1$ and $\|x - y\| \geq \eta$, $\|x + y\| \leq 2(1 - \delta)$. X is called *uniformly convex* if the δ can be chosen independent of x .

PROPOSITION 4.2. *Let X be a locally uniformly convex space. Then every proximal subspace is locally U -proximal.*

PROOF. Let M be a proximal subspace of X and let $x \in X \setminus M$. Let $X_1 = \langle M, x \rangle$ and let $f \in X_1^*$ with $f^{-1}(0) = M$ and $\|f\| = 1$. If $y \in S$ with $f(y) = 1$ and $\alpha(x, \rho)$ is defined as in Proposition 2.2, then

$$\begin{aligned} \alpha(x, \rho) &= d_H((1 + \rho)S[x] \cap f^{-1}(1), \{y\}) \\ &\leq \text{diam}((1 + \rho)S[x] \cap f^{-1}(1)). \end{aligned}$$

By the locally uniformly convexity of the norm on X_1 , we can show that the last term approaches to zero as $\rho \rightarrow 0$. Let $\varepsilon(x, \rho) = \alpha(x, \rho) + \rho$; it follows that M is locally U -proximal. \square

PROPOSITION 4.3. *Let X be a uniformly convex space. Then every closed subspace is U -proximal.*

PROOF. Let $|L|$ denote the length of a line segment L and define

$$\varepsilon(\rho) = \sup\{|L|: L \text{ is in } (1 + \rho)S \setminus S\} + \rho.$$

It is clear that $\varepsilon(\rho) \rightarrow 0$ as $\rho \rightarrow 0$ and $\varepsilon(\rho) \geq \alpha(x, \rho) + \rho$ where $\alpha(x, \rho)$ is defined as in Proposition 2.2. Hence

$$(1 + \rho)S \cap (S + M) \subseteq S + \varepsilon(\rho)S(M)$$

and the proof is completed. \square

Let K be a topological space, let $B(K)$ be the space of bounded continuous functions on K with the supremum norm and let $C(K)$ be the subspace of bounded continuous functions in $B(K)$. It is well known that $C(K)$ is a proximal subspace in $B(K)$ [6].

PROPOSITION 4.4. *Let M be a closed subspace in $X = B(K)$ such that for each $h \in M$, $h \wedge \rho$, $h \vee (-\rho)$, $\rho > 0$, are also in M . Then M satisfies*

$$(1 + \rho)S \cap (S + M) \subseteq S + \rho S(M), \quad \rho > 0.$$

Thus M is U -proximal, and in particular, $C(K)$ is U -proximal.

PROOF. For any $f \in (1 + \rho)S \cap (S + M)$, we can write $f = g + h$ where $\|g\| \leq 1$ and $h \in M$. Let $h' = (h \wedge \rho) \vee (-\rho)$; then $h' \in M$ and $\|h'\| \leq \rho$. Let $g' = f - h'$, we need only show that $\|g'\| \leq 1$ and hence

$$f = g' + h' \in S + \rho S(M).$$

If $|h(x)| \leq \rho$, then $|g'(x)| = |g(x)| \leq 1$. If $h(x) > \rho$, then three cases arise:

(i) if $g(x) \geq 0$, then $|g'(x)| = f(x) - \rho \leq 1$;

(ii) if $g(x) < 0$ and $f(x) \geq 0$, then

$$|g'(x)| = |f(x) - \rho| \leq (1 + \rho) - \rho = 1;$$

(iii) if $g(x) < 0$ and $f(x) < 0$, then $|g(x)| = |f(x)| + h(x)$. This implies $|f(x)| < 1 - \rho$ and

$$|g'(x)| = |f(x) - \rho| \leq |f(x)| + \rho \leq 1.$$

If $h(x) < -\rho$, a similar proof shows that $|g'(x)| \leq 1$. Hence $\|g'\| \leq 1$ and the proof is completed. \square

5. Semi- M -ideals. A convex subset F in a convex set K is a *face* of K if $\lambda y + (1 - \lambda)z \in F, y, z \in K, 0 < \lambda < 1$ implies $y, z \in F$. Let X be a Banach space and let $x \in S$ with $\|x\| = 1$, we use $\text{face}(x)$ to denote the maximal proper face in S containing x .

Let J be a closed subspace of a Banach space X and let

$$J' = \left\{ x \in X: x = 0 \text{ or } \text{face}\left(\frac{x}{\|x\|}\right) \cap J = \emptyset \right\}.$$

J is called a *semi- L -summand* if each $x \in X$ has a unique decomposition as $x = y + z$ with $y \in J, z \in J'$ and $\|x\| = \|y\| + \|z\|$. J is called an *L -summand* if J' is a closed linear subspace of X . For detailed discussion of these and the semi- M -ideals, M -ideals, which we will define later, we refer the readers to [1], [12]. It follows directly from the definition that

$$S = \text{conv}((S \cap J) \cup (S \cap J')).$$

THEOREM 5.1. *Every semi- L -summand J in a Banach space X is U -proximal.*

PROOF. Let $x \in (1 + \rho)S \cap (S + J)$ and $\|x\| = 1 + \rho$; then $x = (1 + \rho)(y + z)$ with $y \in S \cap J$ and $z \in S \cap J'$. Note that $x \in S + J, d(x, J) \leq 1$. This implies $\|(1 + \rho)z\| \leq 1$. Also note that $(1 + \rho) - \rho/\|y\| \geq 0$ (for otherwise, $(1 + \rho)\|y\| < \rho$ will imply $(1 + \rho)\|y\| + (1 + \rho)\|z\| < 1 + \rho$). Let $y' = ((1 + \rho) - \rho/\|y\|)y$ and write $x = (y' + (1 + \rho)z) + \rho y/\|y\|$. Then $\|y' + (1 + \rho)z\| = 1$ and hence $x \in S + \rho(S \cap J)$. \square

Let M be a closed subspace of a Banach space X . M is called a *semi- M -ideal* (*M -ideal*) if $M^\perp = \{x^* \in S^*: x^*(x) = 0 \ \forall x \in M\}$ is a semi- L -summand (L -summand, respectively) in X^* . M is said to have the *n -ball property* if for any $\epsilon > 0$ and for any n intersecting balls $S_i(a_i, r_i) = \{x: \|x - a_i\| \leq r_i\}$ with $S_i(a_i, r_i) \cap M \neq \emptyset$, then $\bigcap_{i=1}^n S_i(a_i, r_i + \epsilon) \cap M \neq \emptyset$. It was proved that M is a semi- M -ideal (M -ideal) if and only if M has the 2-ball (n -ball, $n \geq 3$, respectively) property [1], [12]. In [5], Holmes showed that M -ideals are U -proximal. In the following, we will show that his theorem also holds for semi- M -ideals.

LEMMA 5.2. *Let X be a Banach space and let $e^* \in X^*$ with $\|e^*\| = 1$. Let $F = \{x \in S: e^*(x) = 1\}$. Suppose $S = \text{conv}(F \cup -F)$. Then $M = \{x: e^*(x) = 0\}$ satisfies*

$$(1 + \rho)S \cap (S + M) \subseteq S + \rho S(M).$$

PROOF. Let $x \in (1 + \rho)S \cap (S + M)$ with $\|x\| = 1 + \rho$. By assumption, there exist $a, b \in (1 + \rho)F$ such that $x \in [a : -b]$. Let $y = (\rho/(1 + \rho))((a - b)/2)$. It is easy to show that $x = (x - y) + y \in S + \rho S(M)$. \square

LEMMA 5.3. *Let J be a one dimensional semi- L -summand in X . Let $e \in J$ with $\|e\| = 1$. Then e is an extreme point of S . Moreover, for any $x \in S$, $\|x\| = 1$, at least one of the line segments joining x with e and $-e$ is contained in the boundary of S .*

PROOF. Suppose $e = \frac{1}{2}(x + y)$, $x, y \in S$, $x, y \neq e$. By the definition of semi- L -summand, there exists $x_1, y_1 \in J'$, $0 < \alpha, \beta < 1$ such that

$$x = \alpha x_1 + (1 - \alpha)e, \quad y = \beta y_1 + (1 - \beta)e.$$

Hence $e = (\alpha x_1 + \beta y_1)/(\alpha + \beta)$. This contradicts that J is a semi- L -summand. The second part is clear by observing that the unit sphere of the subspace generated by e and x is a parallelogram. \square

THEOREM 5.4. *Let M be a semi- M -ideal in X . Then M satisfies*

$$(1 + \rho)S \cap \overline{(S + M)} \subseteq \overline{S + \rho S(M)}$$

and hence it is a U -proximal subspace in X .

PROOF. Let $x \in (1 + \rho)S \cap (S + M)$ with $\|x\| = 1 + \rho$. Note that $\langle M, x \rangle$ also has the 2-ball property. We may assume, without loss of generality, that $X = \langle M, x \rangle$. Then $J = M^\perp$ is a one dimensional semi- L -summand in X^* . Let $e^* \in J \cap S$. It follows from Lemma 5.3 and [13, p. 44, Theorem 4.7] that for any extreme point x^{**} in $S(X^{**})$, $x^{**}(e^*) = 1$. Let

$$F = \{x^{**} \in S(X^{**}) : x^{**}(e^*) = 1\};$$

$F \cup -F$ contains all extreme points of $S(X^{**})$. By the Krein-Milman theorem and the fact that F is w^* -compact,

$$\text{conv}(F \cup -F) = \overline{\text{conv}}^{w^*}(F \cup -F) = S(X^{**}).$$

Hence Lemma 5.2 applies and

$$\begin{aligned} x \in (S(X^{**}) + \rho(S(X^{**}) \cap M^{**})) \cap X &= \overline{S(X) + \rho(S(X) \cap M)}^{w^*} \cap S \\ &\subseteq \overline{S(X) + \rho(S(X) \cap M)}. \end{aligned}$$

It follows that $(1 + \rho)S \cap \overline{(S + M)} \subseteq \overline{S + \rho S(M)}$. Proposition 2.4 implies M is a U -proximal subspace with $\epsilon(\rho) = 2\rho$. \square

6. Approximation by compact operators. Let E, F be Banach spaces, we will use $L(E, F)$ ($K(E, F)$) to denote the space of bounded linear operators (compact operators) from E into F . In [4], [11], it is shown that if $E = l^p$, $F = l^q$, $1 < p, q < \infty$, then $K(E, F)$ is an M -ideal in $L(E, F)$. By Theorem 5.4, $K(E, F)$ is actually U -proximal in $L(E, F)$. For the case $E = F = l^1$,

Smith and Ward [16] and Mach and Ward [15] showed that $K(l^1, l^1)$ is not an M -ideal in $L(l^1, l^1)$; however, it is a proximal subspace. In the following, we will consider a more general setting and that $K(l^1, l^1)$ is a U -proximal subspace of $L(l^1, l^1)$ comes as a corollary.

LEMMA 6.1. *Let $(W, \mathfrak{B}, \sigma)$ be a positive measure space and suppose $f, g, h \in L^1(W)$ satisfy $f = g + h$, $\|f\| \leq 1 + \rho$, $\|g\| \leq 1$. Then there exist $g', h' \in L^1(W)$ such that $f = g' + h'$ with $\|g'\| \leq 1$, $\|h'\| \leq 8\rho$ and $|h'(x)| \leq |h(x)|$ for all $x \in W$.*

PROOF. We will assume that $1 < \|f\| \leq 1 + \rho$ (otherwise, we can take $g' = f, h' = 0$) and divide the measure space W into three parts:

$$D_1 = \{x: g(x) \geq 0, h(x) \geq 0\} \cup \{x: g(x) \leq 0, h(x) \leq 0\},$$

$$D_2 = \{x: g(x) \geq 0, h(x) < 0, f(x) \leq 0\}$$

$$\cup \{x: g(x) \leq 0, h(x) > 0, f(x) \geq 0\},$$

and

$$D_3 = \{x: g(x) \geq 0, h(x) < 0, f(x) > 0\}$$

$$\cup \{x: g(x) \leq 0, h(x) > 0, f(x) < 0\}.$$

Let f/D denote the restriction of f to D . It is clear that

$$\|f/D_1\| = \|g/D_1\| + \|h/D_1\|,$$

$$\|f/D_2\| = \|h/D_2\| - \|g/D_2\|,$$

and

$$\|f/D_3\| = \|g/D_3\| - \|h/D_3\|.$$

Three cases arise:

Case (i). If $\|h/D_1\| \geq \rho$, we let $h' = \rho \|h\chi_{D_1}\|^{-1} h\chi_{D_1}$ and let $g' = f - h'$. Then it is easy to check that $\|g'\| \leq 1$, $\|h'\| = \rho$ and $|h'(x)| \leq |h(x)|$ for all $x \in W$.

Case (ii). If $\|h/D_1\| < \rho$ and $\|h/D_2\| < 3\rho$, then we have

$$\|f/D_1\| \leq \rho + \|g/D_1\|, \|f/D_2\| < 3\rho$$

and

$$\|f/D_3\| = \|g/D_3\| - \|h/D_3\|.$$

Thus $1 < \|f\| \leq 1 + 4\rho - \|h/D_3\|$, which implies $\|h/D_3\| < 4\rho$ and hence $\|h\| < 8\rho$. For this case, we let $g' = g, h' = h$.

Case (iii). If $\|h/D_1\| < \rho$ and $\|h/D_2\| \geq 3\rho$, then $\|f/D_2\| \geq \rho$. For otherwise, $\|f/D_2\| < \rho$ implies that $\|g/D_2\| \geq 2\rho$. Thus $\|f/D_2\| < \rho < \|g/D_2\| - \rho$ and it follows that

$$\|f\| \leq (\|g/D_1\| + \rho) + (\|g/D_2\| - \rho) + \|g/D_3\| \leq 1,$$

which contradicts our assumption that $1 < \|f\|$ and proves the claim. We

define $h' = \rho \|f\chi_{D_2}\|^{-1} f\chi_{D_2}$, and let $g' = f - h'$. Then for $x \in D_2$, $|h'(x)| \leq |f(x)| < |h(x)|$ and hence $|h'(x)| < |h(x)|$ for all $x \in W$. It is clear that $\|g'\| = \|f\| - \rho \leq 1$. \square

Let (Ω, Σ, μ) be a σ -finite measure space and let Y be a Banach space. A function $f: \Omega \rightarrow Y$ is called a *Bochner measurable function* if there exists a sequence of functions $\{f_n\}$ of the form $\sum_{i=1}^{\infty} y_i \chi_{E_i}$, where $y_i \in Y$, $\{E_i\}$ is a measurable partition of Ω and $\{f_n\}$ converges to F uniformly except on a zero set. Let $L^\infty(\Omega, Y)$ denote the space of bounded Bochner measurable functions from Ω into Y with norm defined by the essential supremum norm. It is clear from the definition that the set of countable valued Bochner measurable functions is dense in $L^\infty(\Omega, Y)$.

Let (Ω, Σ, μ) be a σ -finite measure space and let $(W, \mathfrak{B}, \sigma)$ be a positive measure space. We use $L_{wc}^\infty(\Omega, L^1(W))$ to denote the subspace of $L^\infty(\Omega, L^1(W))$ consists of those functions f such that $f(\Omega \setminus N)$ is a weak precompact set for some zero set N in Ω .

LEMMA 6.2. *Let $H \in L_{wc}^\infty(\Omega, L^1(W))$ and let $H' \in L^\infty(\Omega, L^1(W))$ such that for almost all $\omega \in \Omega$, $|H'(\omega)| \leq |H(\omega)|$. Then $H' \in L_{wc}^\infty(\Omega, L^1(W))$.*

PROOF. It is well known that for any subset K in $L^1(W, \mathfrak{B}, \sigma)$, K is weakly precompact if and only if it is bounded and for any decreasing sequence $\{E_n\} \subseteq \mathfrak{B}$ such that $\cap E_n = \emptyset$, $\{\int_{E_n} f d\sigma\}$ converges uniformly to zero for all $f \in K$ [2, p. 292, p. 430]. This is also equivalent to $\{|f|: f \in K\}$ is weakly precompact. If $H \in L_{wc}^\infty(\Omega, L^1(W))$, then there exists a zero set N_1 such that $H(\Omega \setminus N_1)$ is weakly precompact. By hypothesis, $|H'(\omega)| \leq |H(\omega)|$ for $\omega \in \Omega \setminus N_2$ where N_2 is a zero set, it follows from the above remark that $H'(\Omega \setminus (N_1 \cup N_2))$ is also weakly precompact; therefore $H' \in L_{wc}^\infty(\Omega, L^1(W))$. \square

THEOREM 6.3. *$L_{wc}^\infty(\Omega, L^1(W))$ is a U -proximal subspace of $L^\infty(\Omega, L^1(W))$.*

PROOF. Let $X = L^\infty(\Omega, L^1(W))$ and let $M = L_{wc}^\infty(\Omega, L^1(W))$, in view of Proposition 2.4(i), it suffices to show that

$$(1 + \rho)S \cap (S_D + M_D) \subseteq S + 8\rho S(M)$$

where S_D and M_D denote the dense subsets of functions with countable values in S and M , respectively. Let $F \in (1 + \rho)S \cap (S_D + M_D)$; we can write $F = G + H$ with $\|G\| \leq 1, H \in M$ and

$$G(\omega)(y) = \sum_{n=1}^{\infty} g_n(y) \cdot \chi_{E_n}(\omega),$$

$$H(\omega)(y) = \sum_{n=1}^{\infty} h_n(y) \cdot \chi_{E_n}(\omega),$$

where $\omega \in \Omega, y \in W$ and $\{E_n\}$ is a measurable partition of E . By Lemma 5.1, for each n , there exist g'_n and h'_n such that $\|g'_n\| \leq 1, \|h'_n\| \leq 8\rho, |h'_n| \leq |h|$ and

$g_n + h_n = g'_n + h'_n$. Define G', H' for $\omega \in \Omega, y \in W$ by

$$G'(\omega)(y) = \sum_{n=1}^{\infty} g'_n(y) \cdot \chi_{E_n}(\omega),$$

$$H'(\omega)(y) = \sum_{n=1}^{\infty} h'_n(y) \cdot \chi_{E_n}(\omega).$$

It follows that $\|G'\| \leq 1, \|H'\| \leq 8\rho$ and $|H'(\omega)| \leq |H(\omega)|, \omega \in \Omega$. Hence $H' \in M$ (Lemma 6.2) and $F = G' + H' \in S + 8\rho S(M)$. \square

Let (Ω, Σ, μ) be a σ -finite measure space and let X be a Banach space with separable dual. For any operator $T: L^1(\Omega) \rightarrow X^*$, there exists a bounded Bochner measurable function $F: \Omega \rightarrow X^*$ such that

$$\langle T(f), x \rangle = \int_{\Omega} f(\omega) \langle F(\omega), x \rangle d\mu(\omega), \quad x \in X,$$

and $\|T\| = \|F\|_{\infty}$. Conversely, for any given bounded Bochner measurable function, we can define an integral operator T with F as kernel [2, p. 506]. We will identify $L(L^1(\Omega), X^*)$ and $L^{\infty}(\Omega, X^*)$. Moreover, it is known that T is weakly compact if and only if the corresponding F is in $L^{\infty}_{wc}(\Omega, X^*)$.

THEOREM 6.4. *Let (Ω, Σ, μ) be a σ -finite measure space. Then $K(L^1(\Omega), l^1)$ is a U -proximal subspace of $L(L^1(\Omega), l^1)$.*

PROOF. As is well known, weak sequential convergence and norm convergence are equivalent in l^1 . Hence the Eberlein-Smulian theorem [2, p. 430] implies that weak compactness and norm compactness in l^1 are identical. Upon identifying $K(L^1(\Omega), l^1)$ with $L^{\infty}_{wc}(\Omega, l^1)$, the result follows from Proposition 5.3. \square

COROLLARY 6.5. *$K(l^1, l^1)$ is a U -proximal subspace of $L(l^1, l^1)$.*

To conclude this section, we remark that little is known about the proximality of the subspace of compact operators on $L^p, 1 \leq p \leq \infty, p \neq 2$ or $C(K)$. In [10] it was proved that if (i) $E = L^1(\mu)$ where μ is a σ -finite measure and F is uniformly convex or (ii) E^* is uniformly convex and $F = C(K)$ for some topological space X , then $K(E, F)$ is a proximal subspace in $L(E, F)$. For operators between l^p and $l^q, 1 \leq p, q \leq \infty$, the only remaining unanswered case is $K(l^{\infty}, l^p), 1 < p \leq \infty$.

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